

# MECHANICS OF MATERIALS

CHAPTER

3

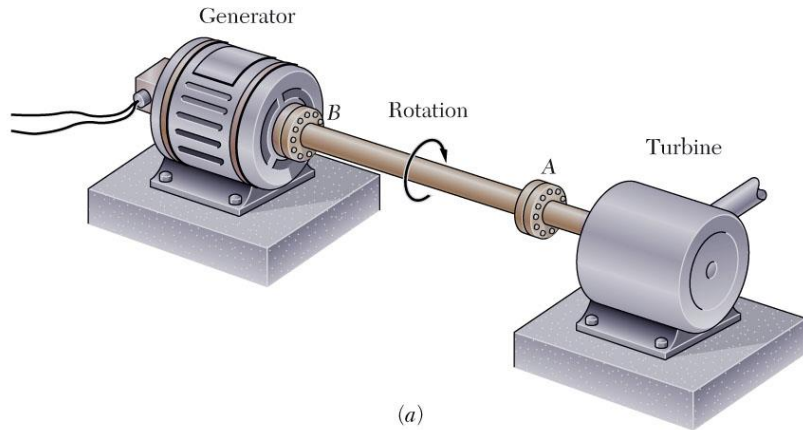
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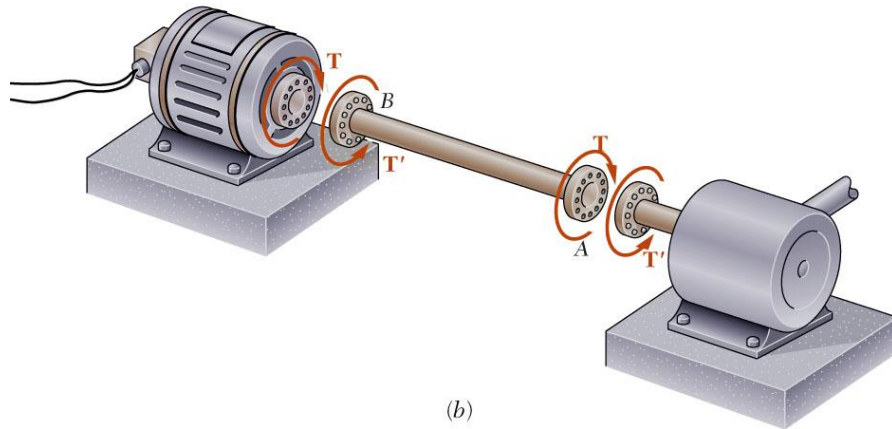
## Torsion



## Torsional Loads on Circular Shafts



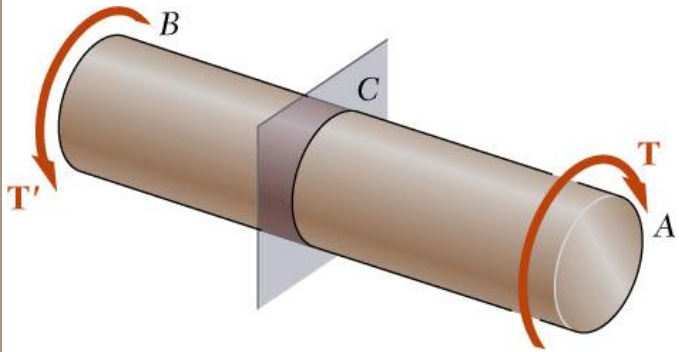
(a)



(b)

- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque  $T$  on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque  $T'$

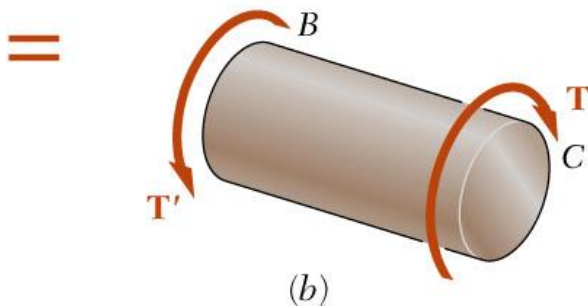
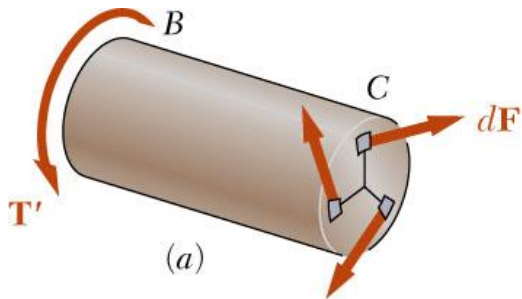
## Net Torque Due to Internal Stresses



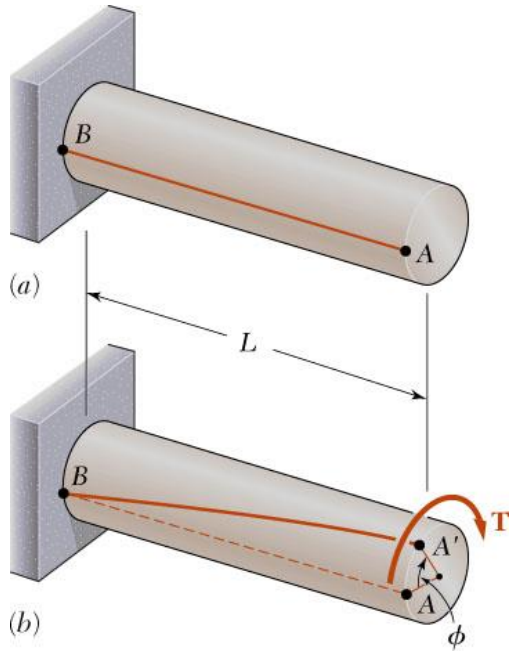
- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho dF = \int \rho(\tau dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not.
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations.
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.



## Shaft Deformations

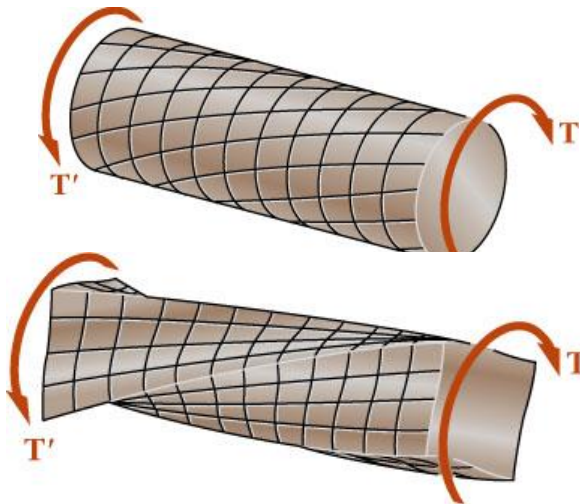


- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

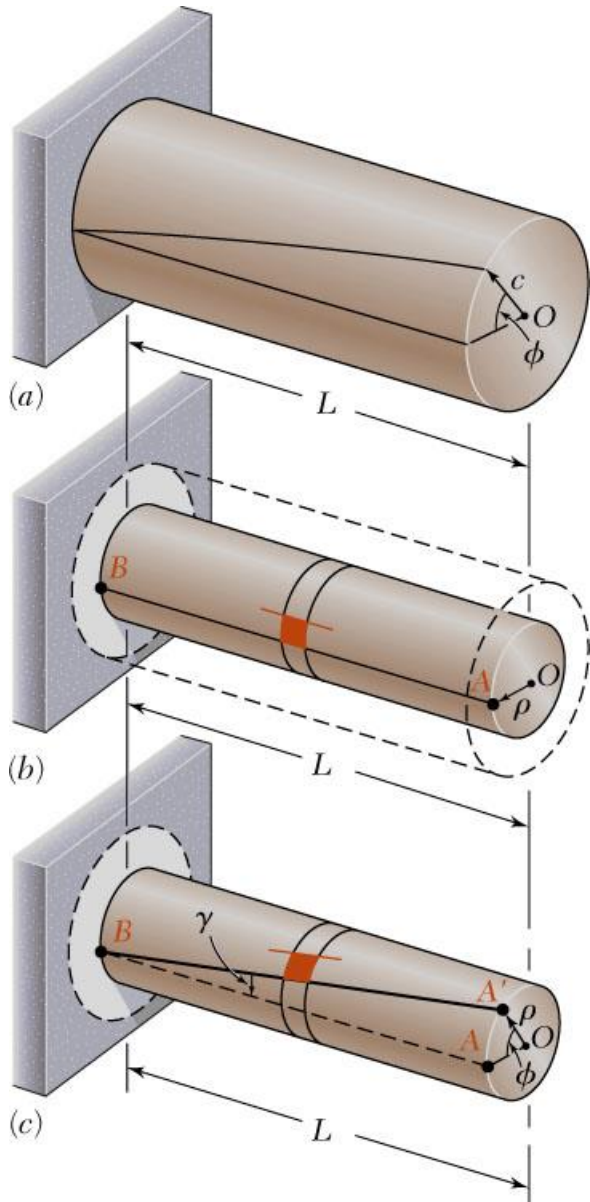
$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.



## Shearing Strain



- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

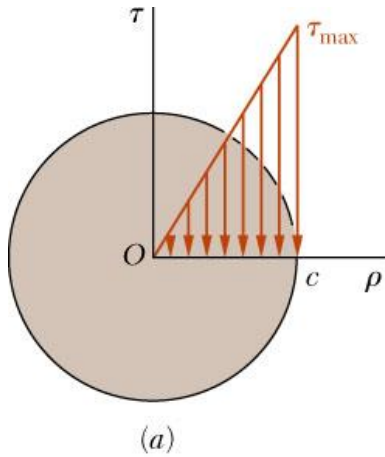
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

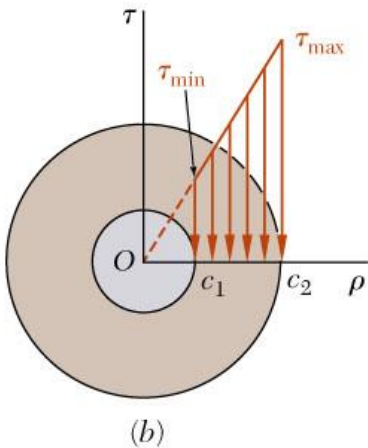
- Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

## Stresses in Elastic Range



$$J = \frac{1}{2} \pi c^4$$



$$J = \frac{1}{2} \pi (c_2^4 - c_1^4)$$

- Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$

From Hooke's Law,  $\tau = G\gamma$ , so

$$\tau = \frac{\rho}{c} \tau_{\max}$$

The shearing stress varies linearly with the radial position in the section.

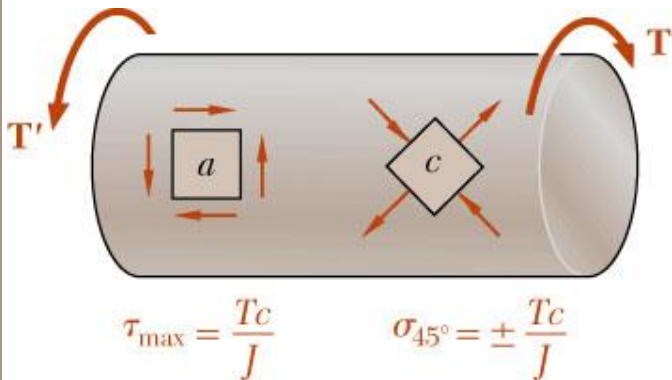
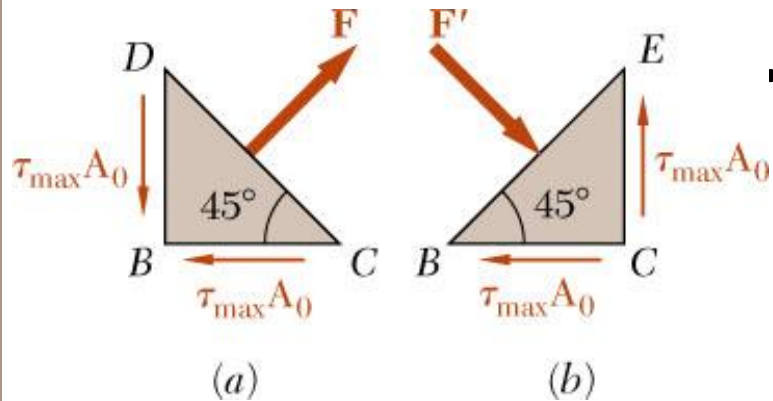
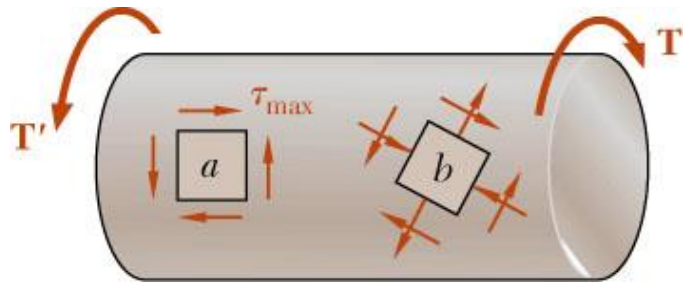
- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho \tau \, dA = \frac{\tau_{\max}}{c} \int \rho^2 \, dA = \frac{\tau_{\max}}{c} J$$

- The results are known as the *elastic torsion formulas*,

$$\tau_{\max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

## Normal Stresses



- Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.

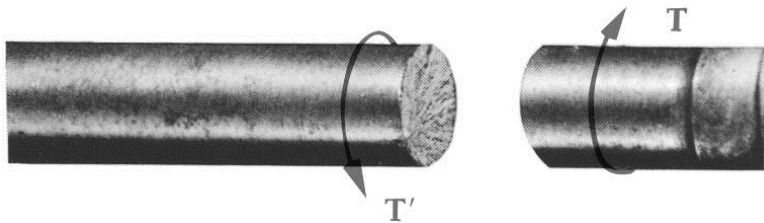
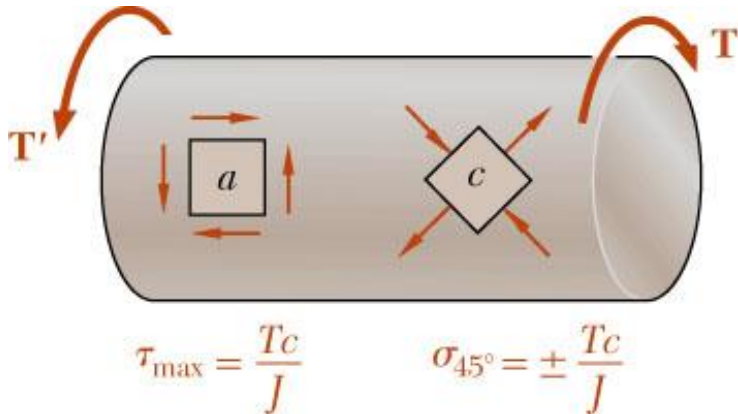
- Consider an element at  $45^\circ$  to the shaft axis,

$$F = 2(\tau_{\max} A_0) \cos 45^\circ = \tau_{\max} A_0 \sqrt{2}$$

$$\sigma_{45^\circ} = \frac{F}{A} = \frac{\tau_{\max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{\max}$$

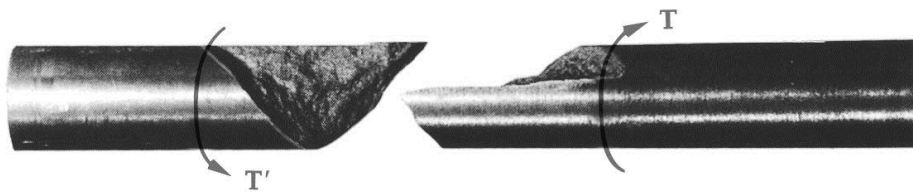
- Element  $a$  is in pure shear.
- Element  $c$  is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements  $a$  and  $c$  have the same magnitude

## Torsional Failure Modes



- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.

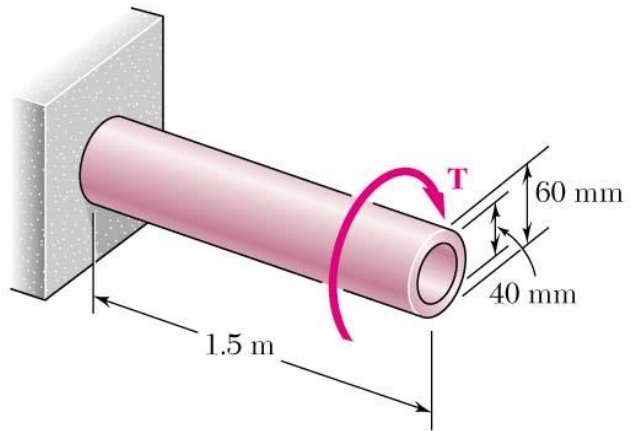
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.



- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at  $45^\circ$  to the shaft axis.

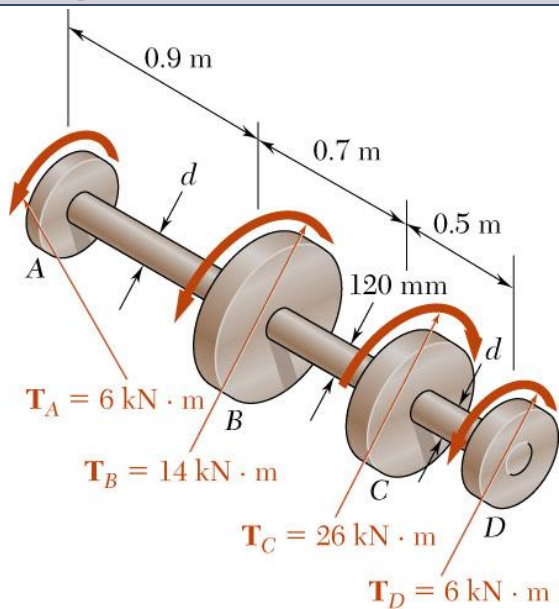


## Sample 3.01



**Fig. 3.16**

## Sample Problem 3.1



Shaft  $BC$  is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts  $AB$  and  $CD$  are solid of diameter  $d$ . For the loading shown, determine (a) the minimum and maximum shearing stress in shaft  $BC$ , (b) the required diameter  $d$  of shafts  $AB$  and  $CD$  if the allowable shearing stress in these shafts is 65 MPa.

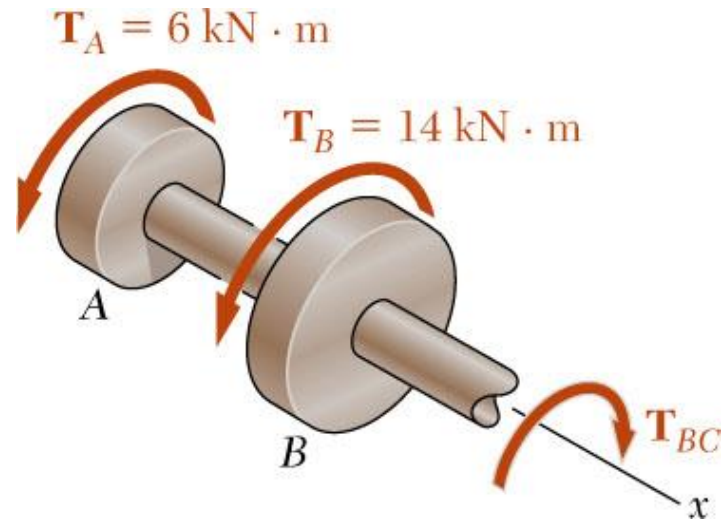
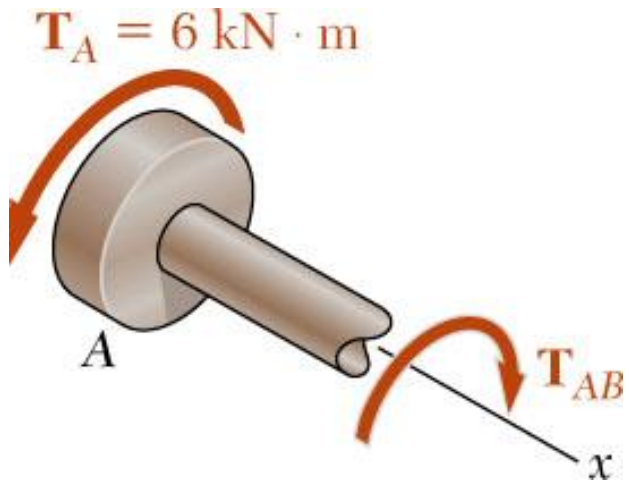
## SOLUTION:

- Cut sections through shafts  $AB$  and  $BC$  and perform static equilibrium analyses to find torque loadings.
- Apply elastic torsion formulas to find minimum and maximum stress on shaft  $BC$ .
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

## Sample Problem 3.1

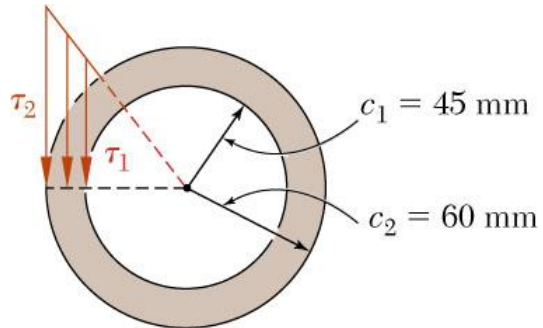
SOLUTION:

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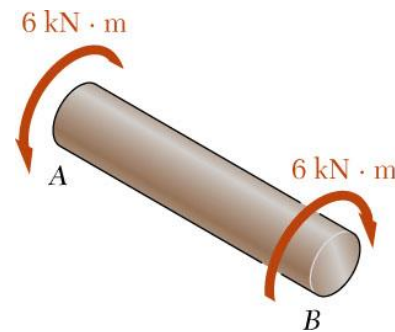


## Sample Problem 3.1

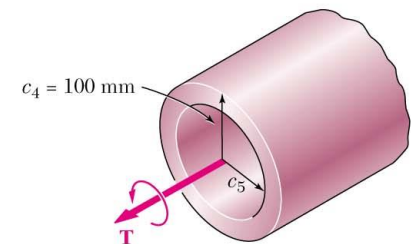
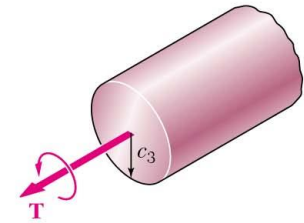
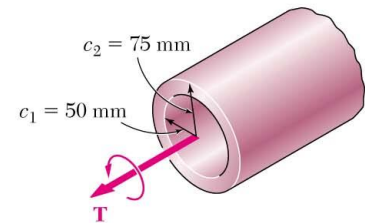
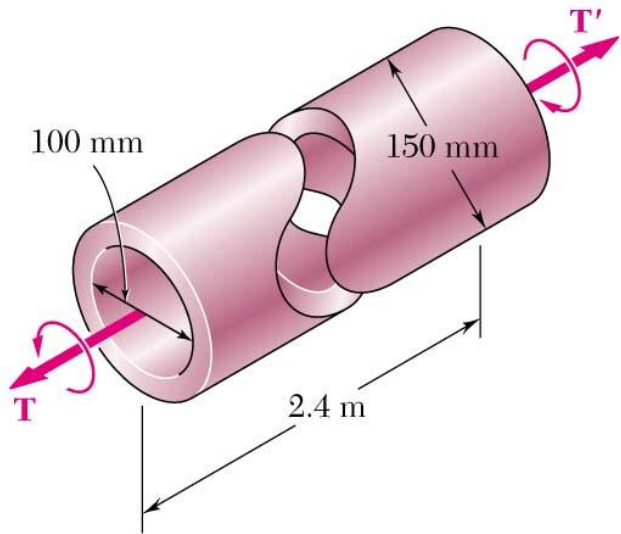
- Apply elastic torsion formulas to find minimum and maximum stress on shaft  $BC$ .



- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.



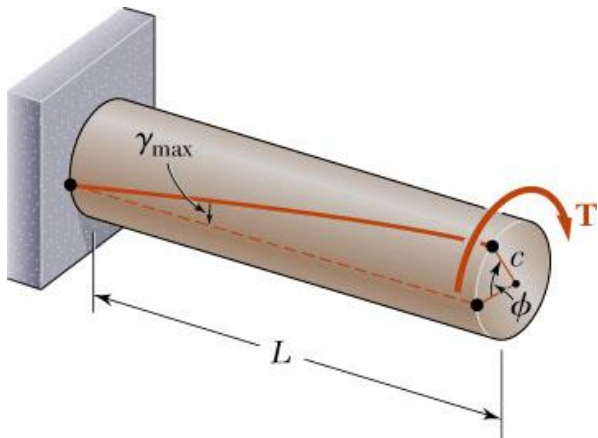
## Sample Problem 3.2



## Problems

- What is moment of inertia?
- 3.10, 3.19, 3.20

## Angle of Twist in Elastic Range



- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{\max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

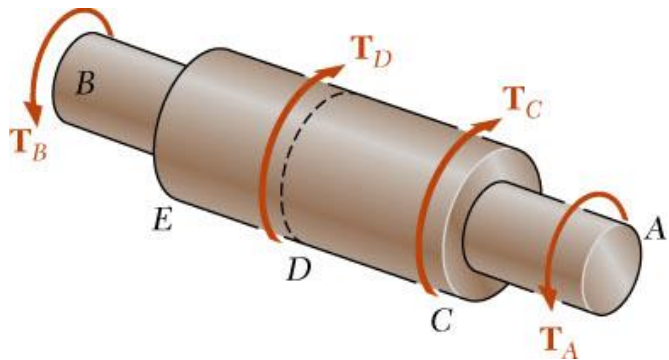
$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG}$$

- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$



## Sample 3.02



## Sample 3.03

## Sample 3.04

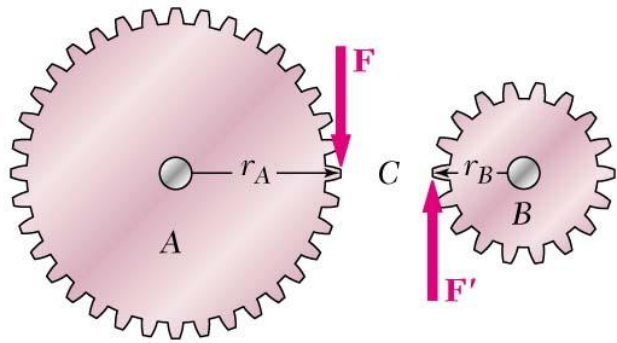
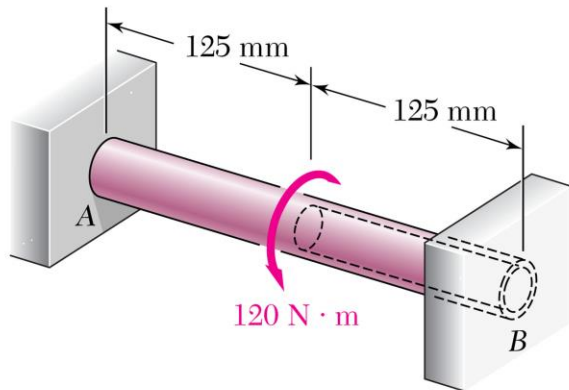


Fig. 3.27

## Statically Indeterminate Shafts



- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.
- From a free-body analysis of the shaft,

$$T_A + T_B = 120 \text{ N} \cdot \text{m}$$

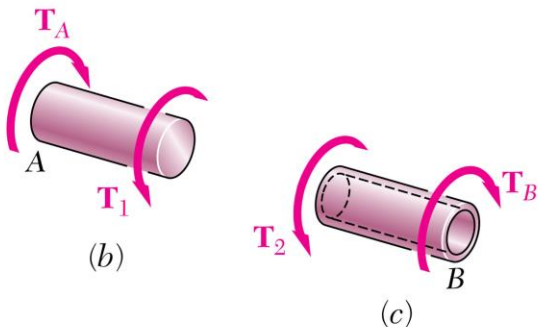
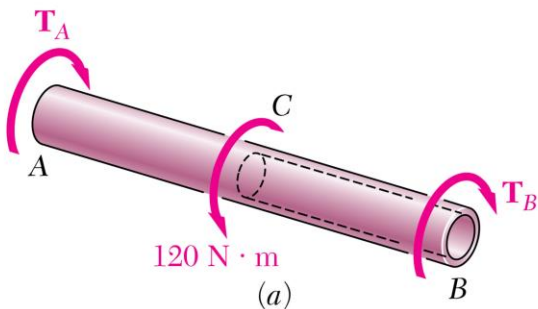
which is not sufficient to find the end torques. The problem is statically indeterminate.

- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \quad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 120 \text{ N} \cdot \text{m}$$



## Sample 3.05

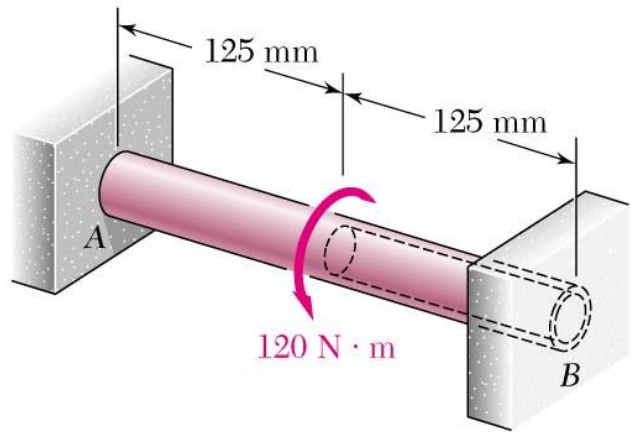


Fig. 3.28

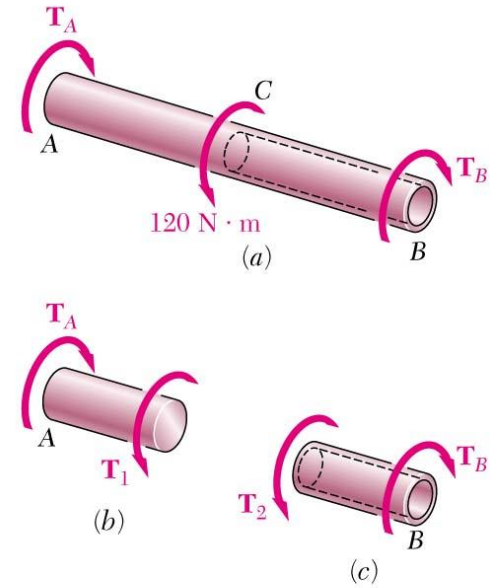
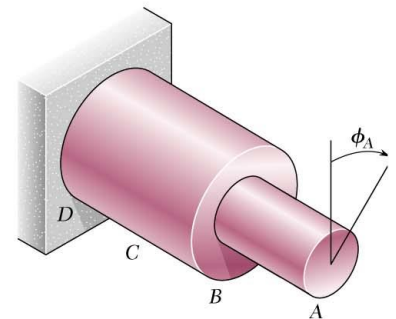
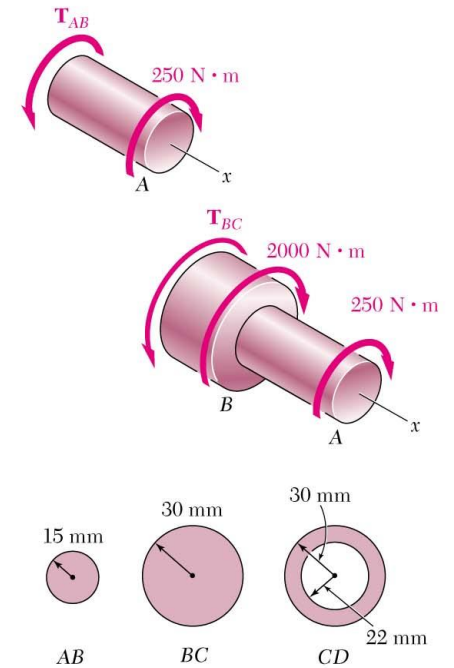
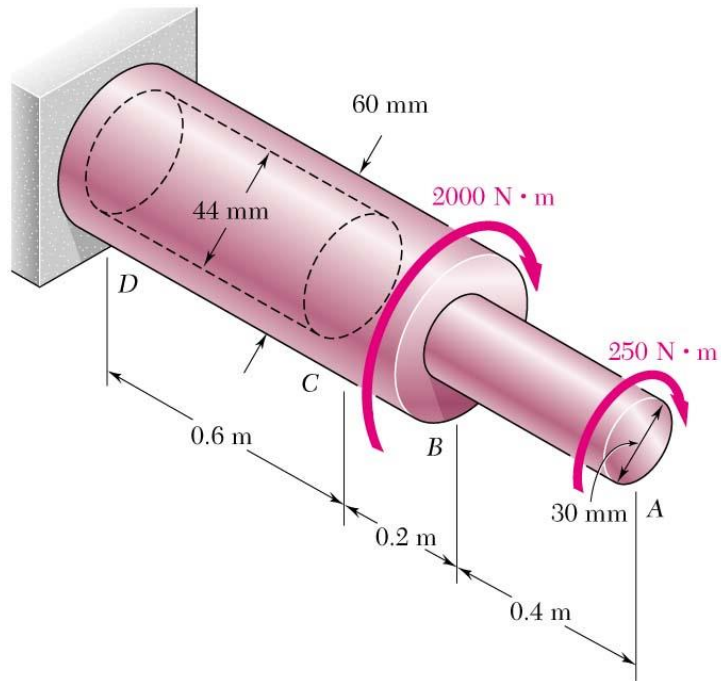


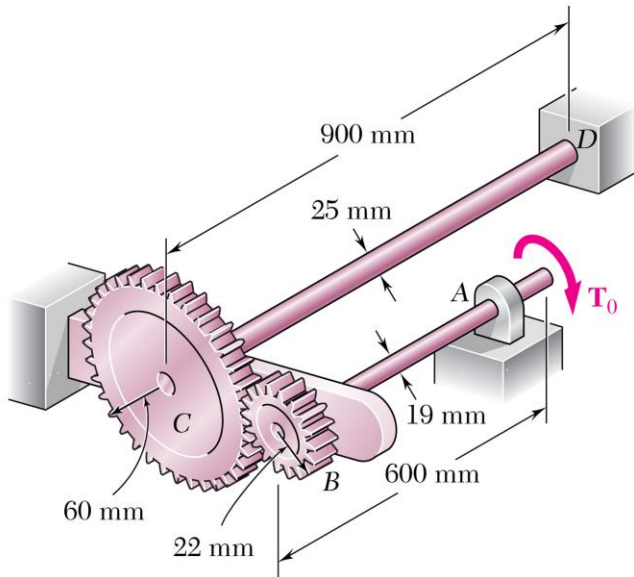
Fig. 3.29

## Sample Problem 3.3



## Sample Problem 3.3

## Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft  $G = 77 \text{ GPa}$  and that the allowable shearing stress is  $55 \text{ MPa}$ , determine (a) the largest torque  $T_0$  that may be applied to the end of shaft  $AB$ , (b) the corresponding angle through which end  $A$  of shaft  $AB$  rotates.

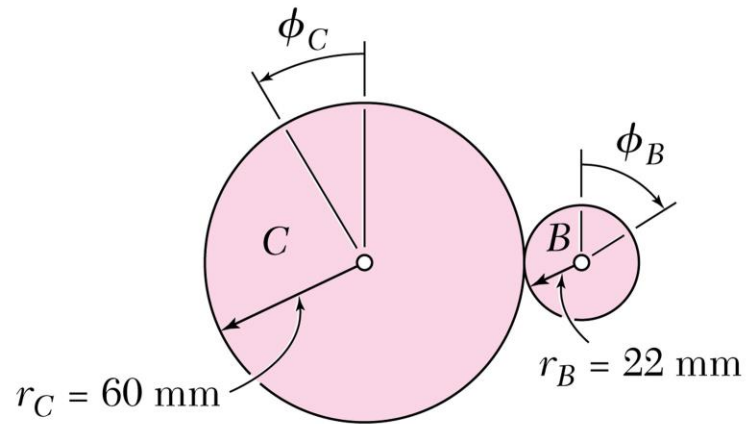
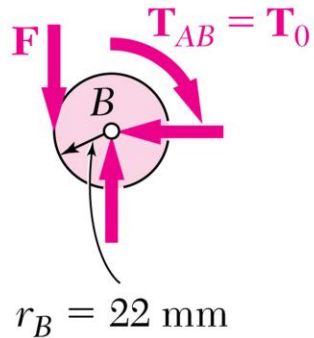
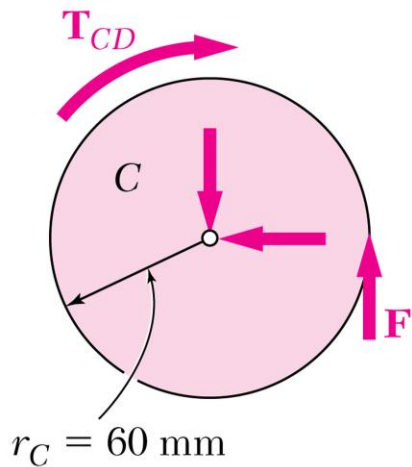
## SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between  $T_{CD}$  and  $T_0$ .
- Apply a kinematic analysis to relate the angular rotations of the gears.
- Find the maximum allowable torque on each shaft – choose the smallest.
- Find the corresponding angle of twist for each shaft and the net angular rotation of end  $A$ .

## Sample Problem 3.4

## SOLUTION:

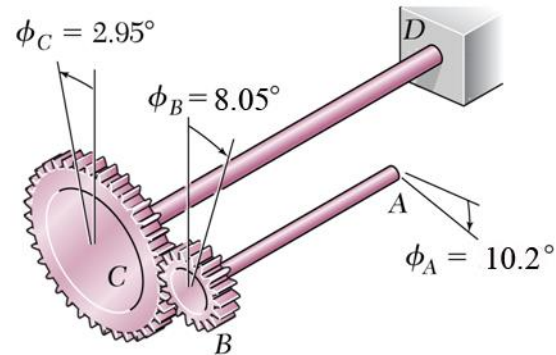
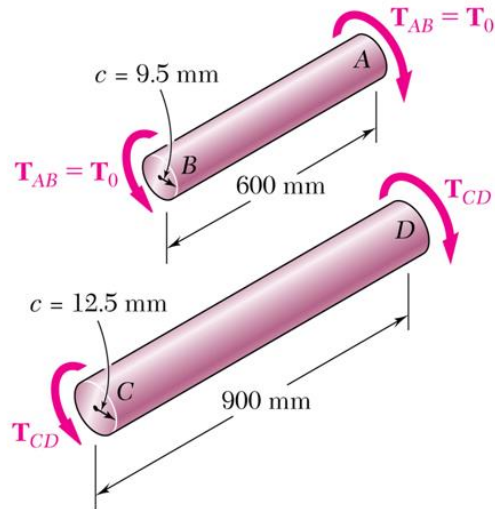
- Apply a static equilibrium analysis on the two shafts to find a relationship between  $T_{CD}$  and  $T_0$ .
- Apply a kinematic analysis to relate the angular rotations of the gears.





## Sample Problem 3.4

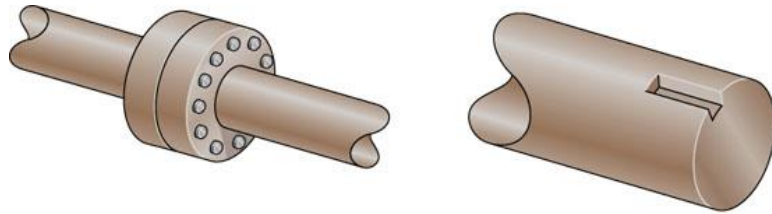
- Find the  $T_0$  for the maximum allowable torque on each shaft – choose the smallest.
- Find the corresponding angle of twist for each shaft and the net angular rotation of end A.



## Problems

- Sample problem 3.5
- 3.41

## Stress Concentrations



- The derivation of the torsion formula,

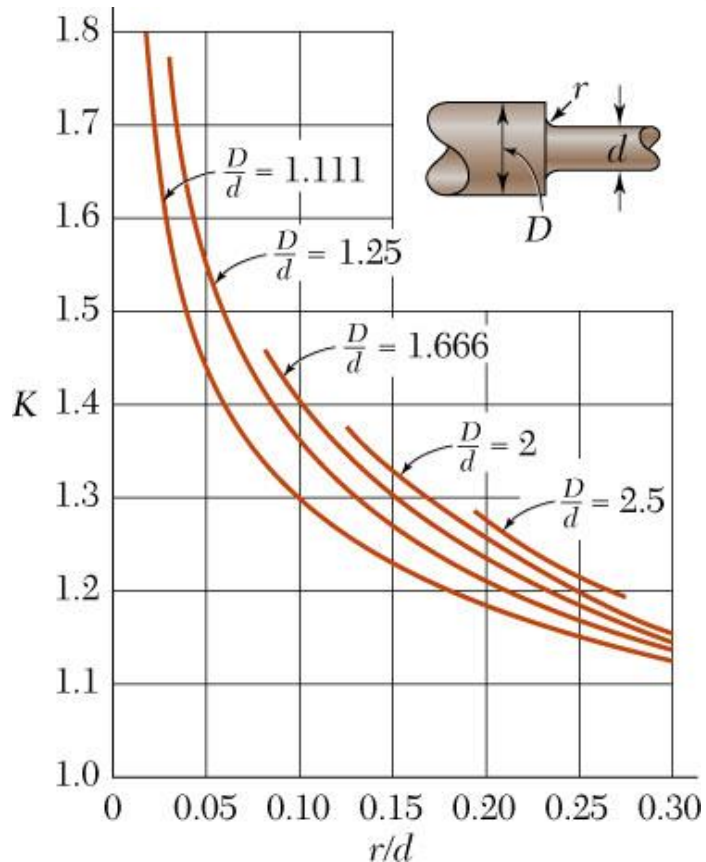
$$\tau_{\max} = \frac{Tc}{J}$$

assumed a circular shaft with uniform cross-section loaded through rigid end plates.

- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

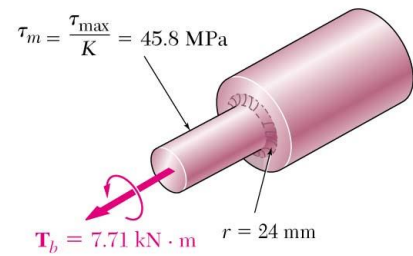
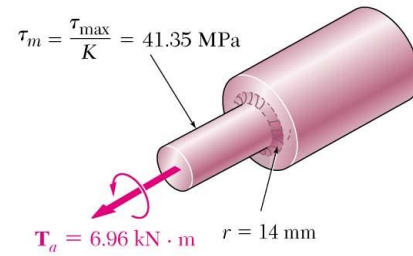
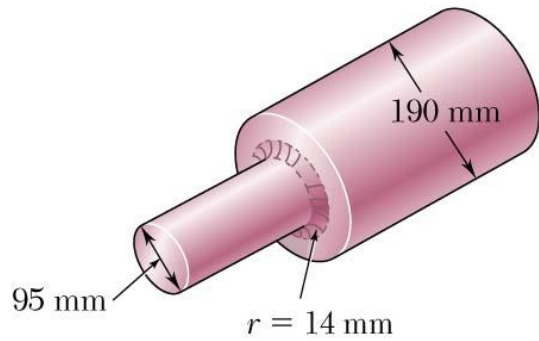
- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K \frac{Tc}{J}$$



**Fig. 3.32** Stress-concentration factors for fillets in circular shafts.

## Sample Problem 3.6



## Problems

- 3.86, 3.87, 3.88